A Three-State Finite Automaton for Carry Propagation in the Accelerated Collatz Map:

Spectral Gap and Closed-Form Shift Dynamics

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Abstract

The accelerated Collatz map $S(n) = (3n+1)/2^{\mathrm{v}_2(3n+1)}$ on the odd integers is governed by the 2-adic valuation $\mathrm{v}_2(3n+1)$, which in turn depends on the binary carry pattern produced when $3n+1$ is computed bit-by-bit. We model this carry propagation by a three-state finite automaton on the state set $\{A, B, C\}$ tracking the joint configuration of the previous input bit and the carry. Our first main result (Theorem 3.3) establishes that the automaton reproduces $3n+1$ exactly and that the transition matrix on uniformly random binary input has characteristic polynomial $(\lambda-1)(2\lambda-1)(2\lambda+1)$, hence eigenvalues $\{1, 1/2, -1/2\}$ and spectral gap 1/2. Our second main result (Proposition 4.1) is a closed-form identity: for an odd integer n with exactly $t \geq 1$ trailing 1-bits, the first $t-1$ accelerated Collatz steps act on the shifted coordinate $x = n+1$ as the pure scaling $x \mapsto (3/2)^{t-1}x$, with no error term. As a corollary we obtain an exponential mixing rate for the carry state under the random-input model and an exact stationary distribution for the carry/previous-bit pair. These results provide a finite-state geometric framework for analyzing the deterministic evolution of v_2 along Collatz orbits, complementing existing 2-adic and ergodic approaches.

Keywords: Collatz map, $3x+1$ problem, finite automaton, carry propagation, spectral gap, 2-adic valuation.

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1 Introduction

The Collatz map $T\colon \mathbb{N} \to \mathbb{N}$ defined by $T(n) = n/2$ for even n and $T(n) = 3n+1$ for odd n has resisted proof of its eponymous conjecture (that every orbit reaches 1) for nearly a century; see the surveys of Lagarias [2, 3] for the history and Tao [4] for the strongest known density result.

For analytical purposes one usually replaces T by the *accelerated* (sometimes called *compressed*) Collatz map restricted to the odd integers,

$$S\colon \mathbb{N}_{\mathrm{odd}} \to \mathbb{N}_{\mathrm{odd}}, \qquad S(n) = \frac{3n+1}{2^{\mathrm{v}_2(3n+1)}}, \tag{1}$$

where $\mathrm{v}_2(m)$ denotes the 2-adic valuation of m. The behavior of S is controlled by the sequence of valuations $\mathrm{v}_2(3n+1), \mathrm{v}_2(3S(n)+1), \ldots$ along an orbit; in particular, an orbit decreases on average if and only if the empirical mean of v_2 exceeds $\log_2 3 \approx 1.585$.

Why carry propagation? The valuation $\mathrm{v}_2(3n+1)$ is an arithmetic quantity determined by the *low-order* bits of $3n+1$, which depend on the low-order bits of n together with the carries that propagate through the binary addition $3n+1 = n + 2n + 1$. Carries are local at every bit position but can propagate arbitrarily far; this dual character is what makes $\mathrm{v}_2(3n+1)$ both deterministic in n and difficult to control as a function of n.

What is known. Several authors have studied 2-adic and automata-theoretic descriptions of the Collatz map. Lagarias [2] treats the parity sequence and shift dynamics on $\mathbb{Z}_2$. Bernstein and Lagarias [1] give a 2-adic conjugacy with the shift map. Wirsching [5] develops the dynamical systems viewpoint and tree models. Various authors have used finite transducers for parity vectors; see, e.g., the survey [3] and references therein. None of these works, to the best of our knowledge, isolates the *three-state* carry automaton studied here or computes its spectral gap.

Our contribution. We make two contributions, both elementary and self-contained.

(i) *Three-state carry automaton (Theorem 3.3).* We construct a finite automaton with three states $\{A, B, C\}$ that tracks the joint configuration (previous input bit, carry) and reproduces the binary representation of $3n + 1$ bit-by-bit. The transition matrix on uniformly random input bits has characteristic polynomial $(\lambda - 1)(2\lambda - 1)(2\lambda + 1)$, hence eigenvalues $\{1, 1/2, -1/2\}$ and spectral gap 1/2. The stationary distribution is the uniform distribution $\pi = (1/3, 1/3, 1/3)$ on $\{A, B, C\}$.

(ii) *Closed-form shift identity (Proposition 4.1).* For an odd integer n with exactly $t \geq 1$ trailing 1-bits in its binary expansion, write $n = 2^t m' - 1$ with m' odd. Then the first $t-1$ accelerated Collatz steps (which are forced to have valuation 1) act on the shifted coordinate $x = n + 1$ exactly as

$$x \longmapsto \left(\tfrac{3}{2}\right)^{t-1} x,$$

with no error term, so that $S^{t-1}(n) = (3/2)^{t-1}(n + 1) - 1 = 2 \cdot 3^{t-1} m' - 1$.

These two results combine in Corollary 3.4 to give an exponential mixing rate of order $(1/2)^t$ for the carry state under the random-input model, and a clean description of the deterministic "trailing-1 epoch" as a pure scaling in shifted coordinates.

What we do *not* claim. The random-input model is a heuristic device. The orbits of S generate inputs to the automaton that are highly correlated with the state itself, and our spectral gap does *not* imply mixing along deterministic orbits. In particular, the present paper does not address the Collatz conjecture; it provides a clean finite-state geometric description of one of its building blocks. See Section 6 for a careful discussion.

Paper structure. Section 2 fixes notation and recalls the binary arithmetic of $3n+1$. Section 3 constructs the three-state automaton and proves Theorem 3.3. Section 4 proves the closed-form shift identity (Proposition 4.1) and derives Corollary 3.4. Section 5 illustrates the framework on small examples. Section 6 discusses connections to existing approaches and limitations.

2 Preliminaries

Throughout, $\mathbb{N}_{\text{odd}} = \{1, 3, 5, \dots\}$ denotes the positive odd integers and $\mathrm{v}_2(m)$ the 2-adic valuation of $m \in \mathbb{Z} \setminus \{0\}$, i.e. the largest k with $2^k \mid m$. We write $b_i(n) \in \{0, 1\}$ for the i-th bit of n in its binary expansion, so that $n = \sum_{i \geq 0} b_i(n)\, 2^i$.

Definition 2.1 (Trailing 1-bit count)**.** For an odd integer $n \geq 1$ we set

$$t(n) \;=\; \mathrm{v}_2(n + 1) \;\geq\; 1,$$

the number of trailing 1-bits of n in binary. Equivalently, $t(n)$ is the unique $t \geq 1$ such that $n \equiv 2^t - 1 \pmod{2^{t+1}}$. We write $n = 2^t m' - 1$ with $m' \in \mathbb{N}$ odd; this representation is unique.

Definition 2.2 (Accelerated and one-step maps)**.** The *accelerated Collatz map* $S\colon \mathbb{N}_{\mathrm{odd}} \to \mathbb{N}_{\mathrm{odd}}$ is given by (1). We also use the *one-step half map*

$$T\colon \mathbb{N}_{\mathrm{odd}} \to \mathbb{N}, \qquad T(n) = (3n+1)/2,$$

defined for every odd n (since $3n+1$ is always even). The image $T(n)$ is odd precisely when $\mathrm{v}_2(3n+1) = 1$, in which case $T(n) = S(n)$.

Lemma 2.3. *For every odd integer* $n \geq 1$*,*

$$\mathrm{v}_2(3n+1) \;=\; 1 \quad\iff\quad n \equiv 3 \pmod 4,$$

and equivalently $\mathrm{v}_2(3n+1) \geq 2$ *iff* $n \equiv 1 \pmod 4$*.*

Proof. If $n \equiv 3 \pmod 4$, write $n = 4k+3$; then $3n+1 = 12k+10 = 2(6k+5)$, and $6k+5$ is odd, so $\mathrm{v}_2(3n+1) = 1$. If instead $n \equiv 1 \pmod 4$, write $n = 4k+1$; then $3n+1 = 12k+4 = 4(3k+1)$, so $\mathrm{v}_2(3n+1) \geq 2$. Together these give the stated equivalence. ☐

Lemma 2.4. *Let* $n \geq 1$ *be odd with* $t = t(n) \geq 1$ *trailing* 1*-bits. Then* $T(n) = (3n+1)/2$ *is an odd integer if and only if* $n \equiv 3 \pmod 4$*, which holds if and only if* $t(n) \geq 2$*.*

Proof. $T(n) \in \mathbb{Z}$ for every odd n, since $3n+1$ is even. We have $T(n)$ odd iff $\mathrm{v}_2(3n+1) = 1$ iff $n \equiv 3 \pmod 4$ (Lemma 2.3), iff $b_0(n) = b_1(n) = 1$, iff $t(n) \geq 2$. ☐

Definition 2.5 (Bad-step segment)**.** Let n be odd with $t = t(n) \geq 2$. A *bad step* is an application of T for which $\mathrm{v}_2(3n+1) = 1$, equivalently the output is again odd. We call the run of $t-1$ consecutive bad steps beginning at n the *bad-step segment* of length $t-1$ at n.

The terminology reflects that bad steps produce no 2-adic contraction beyond the forced single division by 2, and hence contribute $\log_2(3/2) \approx 0.585$ of growth (in $\log_2$ scale) each.

3 The Three-State Carry Automaton

We now construct a finite automaton that reproduces the binary expansion of $3n+1$ bit-by-bit and analyze its spectrum.

3.1 Construction

Computing $3n+1 = n+2n+1$ in binary by elementary school addition proceeds bit-by-bit from the least significant position. At position $i \geq 0$ we add three contributions: the bit $b_i(n)$ from the first copy of n, the bit $b_{i-1}(n)$ from the shifted copy $2n$ (with the convention $b_{-1}(n) = 0$), the constant offset $\delta_{i,0} \in \{0,1\}$ (equal to 1 for $i = 0$ and 0 otherwise), and an incoming carry $c_{i-1} \in \{0,1\}$ from position $i-1$ (with $c_{-1} = 0$).

Writing $\sigma_i = b_i(n) + b_{i-1}(n) + \delta_{i,0} + c_{i-1}$, the output bit and outgoing carry are

$$o_i \;=\; \sigma_i \bmod 2, \qquad c_i \;=\; \lfloor \sigma_i/2 \rfloor. \tag{2}$$

Since each summand is in $\{0,1\}$, $\sigma_i \in \{0,1,2,3\}$ and $c_i \in \{0,1\}$.

State. The information that the bit-by-bit computation must carry forward from position $i-1$ to position i consists of

- the previous input bit $b_{i-1}(n)$ (to be used as the bit of the shifted summand at position i), and
- the carry c_{i-1}.

Both lie in $\{0,1\}$, giving four configurations. Examining (2) shows that for $i \geq 1$ the configurations $(b_{i-1}, c_{i-1}) = (1,0)$ and $(0,1)$ produce identical (o_i, c_i) behavior as functions of the next input bit $b_i \in \{0,1\}$:

$$(1,0)\colon\ \sigma_i = b_i + 1,\ (o_i, c_i) = \begin{cases} (1,0) & b_i = 0 \\ (0,1) & b_i = 1 \end{cases}; \qquad (0,1)\colon\ \sigma_i = b_i + 1,\ \text{same outputs.}$$

Merging these two equivalent configurations yields a three-state automaton with state set $\{A, B, C\}$:

$$A = (0,0), \qquad B = \{(1,0),(0,1)\}, \qquad C = (1,1). \tag{3}$$

The transitions, computed directly from (2), are tabulated in Table 1. The output bit o_i is recorded along each transition arrow.

state	input bit b_i	next state	output o_i
A	0	A	0
A	1	B	1
B	0	A	1
B	1	C	0
C	0	B	0
C	1	C	1

Table 1: Transition table of the three-state carry automaton for $3n+1$. States are defined in (3). The initial state at position $i = 0$ is B, encoding $(b_{-1}, c_{-1}) = (0,1)$, where the virtual carry $c_{-1} = 1$ subsumes the constant offset $\delta_{0,0} = 1$; see Remark 3.1.

Remark 3.1. At position $i = 0$ the constant offset $\delta_{0,0} = 1$ is absorbed into the virtual incoming carry $c_{-1} = 1$, and $b_{-1}(n) = 0$. This is the configuration $(0,1)$, which under the identification (3) corresponds to state B. Hence the initial state of the automaton (before reading b_0) is B. At position $i \geq 1$ the offset contribution vanishes and the table acts unmodified. See Example 5.3.

3.2 Correctness

Lemma 3.2 (Automaton correctness)**.** *Fix an odd integer $n \geq 1$ with binary expansion $(b_0, b_1, b_2, \dots)$ and let $L \geq 1$ with $b_i = 0$ for all $i \geq L$. Driving the automaton of Table 1 with input $b_0, b_1, \dots, b_L$ starting from state B produces the output bits $(o_0, o_1, \dots, o_L)$, which form the binary expansion of $3n+1$ in positions 0 through L.*

Proof. Induction on the position $i \geq 0$. By construction the state after processing input bits $b_0, \dots, b_i$ encodes the pair (b_i, c_i) (modulo the equivalence (3)), where c_i is the outgoing carry from position i in the schoolbook addition $n + 2n + 1$. The output bit o_i is given by (2), which is exactly what the table records. The base case $i = 0$ uses the initial state B identified in Remark 3.1: from B with input $b_0 = 1$ (since n is odd) the table gives output $o_0 = 0$ and next state $C = (1,1)$, matching the schoolbook computation $\sigma_0 = 1 + 0 + 1 + 0 = 2$ which produces output bit 0 and outgoing carry 1. ☐

3.3 Spectrum

We now consider the automaton driven by independent uniformly random input bits $b_i \in \{0,1\}$, ignoring the deterministic structure of the bits of n. The associated transition matrix on $\{A, B, C\}$ is

$$M = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}, \tag{4}$$

where the (j,k)-entry is the probability that the next state is k given that the current state is j, with rows and columns indexed in the order A, B, C.

Theorem 3.3 (Three-state carry automaton; main result)**.** *The matrix M in* (4) *satisfies*

$$\det(\lambda I - M) \;=\; (\lambda-1)\bigl(\lambda-\tfrac{1}{2}\bigr)\bigl(\lambda+\tfrac{1}{2}\bigr).$$

Hence its eigenvalues are $\{1,\, 1/2,\, -1/2\}$, the spectral gap is $1/2$, and its unique stationary distribution on $\{A, B, C\}$ is $\pi = (1/3, 1/3, 1/3)$.

Proof. Expanding the determinant:

$$\begin{aligned}\det(\lambda I - M) &= \det\begin{pmatrix}\lambda-1/2 & -1/2 & 0\\ -1/2 & \lambda & -1/2\\ 0 & -1/2 & \lambda-1/2\end{pmatrix}\\ &= (\lambda-1/2)\bigl[\lambda(\lambda-1/2)-1/4\bigr] - (-1/2)\bigl[-\tfrac{1}{2}(\lambda-1/2)-0\bigr]\\ &= (\lambda-1/2)\bigl(\lambda^2-\lambda/2-1/4\bigr) - \tfrac{1}{4}(\lambda-1/2)\\ &= (\lambda-1/2)\bigl(\lambda^2-\lambda/2-1/2\bigr).\end{aligned}$$

The quadratic factor satisfies $\lambda^2 - \lambda/2 - 1/2 = (\lambda-1)(\lambda+1/2)$, since $1\cdot(-1/2) = -1/2$ and $1 + (-1/2) = 1/2$. Hence

$$\det(\lambda I - M) = (\lambda-1/2)(\lambda-1)(\lambda+1/2),$$

which is the claimed factorization.

The eigenvalues are therefore $\{1, 1/2, -1/2\}$. Since M is a row stochastic matrix, $\lambda = 1$ is the Perron eigenvalue with right eigenvector $(1,1,1)^\top$. A direct computation gives $\pi M = \pi$ for $\pi = (1/3, 1/3, 1/3)$, and uniqueness follows from irreducibility (the transition graph is strongly connected by inspection of Table 1). The spectral gap is $1 - \max\{|1/2|, |-1/2|\} = 1/2$. ☐

Corollary 3.4 (Carry mixing under random input)**.** *Let $\mu_t \in \mathbb{R}^3$ be the distribution of the automaton state after processing t independent uniformly random input bits, starting from any initial distribution μ_0. Then for every $t \geq 0$,*

$$\|\mu_t - \pi\|_1 \;\leq\; 2\cdot(1/2)^t,$$

where $\|\cdot\|_1$ is the total variation (i.e. ℓ^1) norm and $\pi = (1/3, 1/3, 1/3)$.

Proof. Write $\mu_0 - \pi = \alpha v_{1/2} + \beta v_{-1/2}$ with v_λ right eigenvectors of $M^\top$ for eigenvalue λ, noting that $\mu_0 - \pi$ lies in the orthogonal complement of $(1,1,1)$ since both μ_0 and π are probability vectors. Then $\mu_t - \pi = (\mu_0 - \pi)M^t = \alpha(1/2)^t v_{1/2} + \beta(-1/2)^t v_{-1/2}$, so $\|\mu_t - \pi\|_1 \leq (1/2)^t\,\|\mu_0 - \pi\|_1 \leq 2(1/2)^t$. ☐

4 Closed-Form Shift Identity for Trailing-1 Epochs

We now turn to the deterministic side: the first $t-1$ steps of S starting from an odd integer with t trailing 1-bits. Recall from Lemma 2.3 that $\mathrm{v}_2(3n+1) = 1$ when $n \equiv 3 \pmod 4$, which holds throughout a bad-step segment of length $t-1$ (Definition 2.5).

4.1 The shift identity

Proposition 4.1 (Closed-form shift identity)**.** *Let n be an odd positive integer with $t = t(n) \geq 1$ trailing 1-bits, and write $n = 2^t m' - 1$ with $m' \in \mathbb{N}$ odd. Set $x = n + 1 = 2^t m'$. Then $T^{t-1}(n)$ is well-defined as an odd integer and*

$$T^{t-1}(n) + 1 \;=\; \left(\tfrac{3}{2}\right)^{t-1} \cdot x, \qquad \textit{equivalently} \qquad T^{t-1}(n) \;=\; 2 \cdot 3^{t-1} m' - 1. \tag{5}$$

Moreover, each of the iterates $T^j(n)$ for $0 \leq j \leq t-1$ is odd, so $T^{t-1}(n) = S^{t-1}(n)$ as elements of $\mathbb{N}_{\text{odd}}$.

Proof. We prove by induction on j that for $0 \leq j \leq t-1$,

$$T^j(n) \;=\; \frac{3^j n + (3^j - 2^j)}{2^j}. \tag{6}$$

The base case $j = 0$ is the identity $n = n$. For the inductive step, assume (6) holds at level $j < t - 1$. Then

$$T^{j+1}(n) \;=\; T(T^j(n)) \;=\; \frac{3 \cdot \frac{3^j n + (3^j - 2^j)}{2^j} + 1}{2} \;=\; \frac{3^{j+1} n + 3(3^j - 2^j) + 2^j}{2^{j+1}}.$$

The numerator simplifies: $3 \cdot 3^j - 3 \cdot 2^j + 2^j = 3^{j+1} - 2 \cdot 2^j = 3^{j+1} - 2^{j+1}$, giving

$$T^{j+1}(n) = \frac{3^{j+1} n + (3^{j+1} - 2^{j+1})}{2^{j+1}},$$

the claim at level $j+1$. Thus (6) holds for all $0 \leq j \leq t-1$ provided each intermediate division by 2 yields an integer; we now verify this.

Substituting $n = 2^t m' - 1$ into (6):

$$T^j(n) = \frac{3^j(2^t m' - 1) + 3^j - 2^j}{2^j} \;=\; \frac{2^t \cdot 3^j m' - 2^j}{2^j} \;=\; 2^{t-j} \cdot 3^j m' - 1.$$

For $0 \leq j \leq t-1$ the exponent $t - j \geq 1$, so $T^j(n)$ is an odd integer (being one less than an even integer). Thus the formula (6) is valid throughout the bad-step segment, and at $j = t-1$ yields

$$T^{t-1}(n) = 2 \cdot 3^{t-1} m' - 1,$$

which is (5). The first form $T^{t-1}(n) + 1 = (3/2)^{t-1} x$ follows from $x = 2^t m'$: indeed $2 \cdot 3^{t-1} m' = 3^{t-1} \cdot 2m' = (3/2)^{t-1} \cdot 2^t m' \cdot 2/2 = (3/2)^{t-1} x$. Finally, since each $T^j(n)$ is odd for $0 \leq j \leq t-1$, we have $\mathrm{v}_2(3T^j(n) + 1) \geq 1$ (forced) and indeed equal to 1 for $0 \leq j \leq t-2$ by Lemma 2.3 applied to $T^j(n) \equiv 3 \pmod 4$ (verified from $T^j(n) = 2^{t-j} \cdot 3^j m' - 1$ with $t - j \geq 2$ and $3^j m'$ odd). Hence each T-step in the segment coincides with an S-step, so $T^{t-1}(n) = S^{t-1}(n)$. ☐

Remark 4.2. Proposition 4.1 is folkloric in the sense that the formula $T^j(n) = (3^j n + (3^j - 2^j))/2^j$ appears, in various guises, in [2, 5] and elsewhere. The contribution recorded here is the *shift-coordinate* interpretation $x = n + 1$, in which the $t - 1$ deterministic bad steps act as the pure multiplicative scaling $x \mapsto (3/2)^{t-1} x$ with *no* error term. This identity is what makes the bad-step segment exactly analyzable, and underlies the epoch decomposition discussed in [2].

5 Examples

Example 5.1. Take $n = 7 = 111_2$, so $t(7) = 3$ and $m' = 1$. Proposition 4.1 predicts

$$T^2(7) \;=\; 2 \cdot 3^2 \cdot 1 - 1 \;=\; 17,$$

or in shift coordinates $T^2(7) + 1 = 18 = (9/4) \cdot 8 = (3/2)^2 \cdot 8$. Direct computation: $T(7) = 11$, $T(11) = 17$. ✓

Example 5.2. Take $n = 31 = 11111_2$, so $t(31) = 5$ and $m' = 1$. Then $T^4(31) = 2 \cdot 3^4 - 1 = 161$. Direct computation: $T(31) = 47, T(47) = 71, T(71) = 107, T(107) = 161$. ✓

Example 5.3 (Automaton trace for $n = 13$)**.** We have $n = 13 = 1101_2$, so the input bits read from least to most significant are $(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (1, 0, 1, 1, 0, 0, 0)$. Starting in state B, the trace from Table 1 is

$$B \xrightarrow{b_0=1,\, o_0=0} C \xrightarrow{b_1=0,\, o_1=0} B \xrightarrow{b_2=1,\, o_2=0} C \xrightarrow{b_3=1,\, o_3=1} C \xrightarrow{b_4=0,\, o_4=0} B \xrightarrow{b_5=0,\, o_5=1} A \xrightarrow{b_6=0,\, o_6=0} A.$$

Reading the output bits with weight 2^i gives $\sum_{i=0}^{6} o_i \cdot 2^i = 8 + 32 = 40 = 101000_2$, and indeed $3 \cdot 13 + 1 = 40$. ✓

Example 5.4 (Confirming the spectral gap numerically)**.** Starting from the deterministic distribution $\mu_0 = (1, 0, 0)$ (state A), the iterates $\mu_t = \mu_0 M^t$ approach $\pi = (1/3, 1/3, 1/3)$ at the rate guaranteed by Corollary 3.4:

$$\mu_1 = (1/2, 1/2, 0), \quad \mu_2 = (1/2, 1/4, 1/4), \quad \mu_3 = (3/8, 3/8, 1/4), \quad \ldots$$

A short calculation gives $\|\mu_t - \pi\|_1 = (1/2)^t \cdot 4/3$ in this case, well within the $2 \cdot (1/2)^t$ envelope of Corollary 3.4.

6 Discussion

Relation to existing automata-theoretic models. The Collatz parity vector $\big([T^k(n) \bmod 2]\big)_{k \geq 0}$ has been studied extensively as a shift-invariant quantity on $\mathbb{Z}_2$; see [2, 1]. The transducer viewpoint underlying the parity vector typically encodes the *state of the orbit* (the iterate modulo a small power of 2) rather than the *state of the carry computation* for a single application of $3n + 1$. The three-state automaton constructed here is of the latter kind: it operates on the bits of a fixed odd integer n and produces the bits of $3n + 1$ in one pass. The 1/2 spectral gap is intrinsic to this carry computation and does not appear, to our knowledge, in the $\mathbb{Z}_2$-shift literature.

Limits of the random-input model. Theorem 3.3 and its corollary describe the *random* input model: the input bits b_i are independent and uniform. In any deterministic Collatz orbit the bits processed at successive steps are not random; in particular, an adversarial input string could in principle keep the automaton in a single state indefinitely (the self-loops $A \xrightarrow{0} A$ and $C \xrightarrow{1} C$ make this evident). Consequently, the spectral gap does *not* immediately yield mixing along Collatz orbits, and we make no such claim.

Interaction with the shift identity. Proposition 4.1 identifies a deterministic regime in which the bits of the input *are* highly structured: throughout a bad-step segment of length $t - 1$ at $n = 2^t m' - 1$, the automaton is processing input bit $b_0(n) = 1$ at the start (driving $C \to C$) and exhibits behavior tied tightly to the trailing block of 1s. At the end of the segment the bits encountered transition into the binary representation of m', which is unconstrained. The interplay between these two regimes—deterministic block-processing during a segment, followed by entry into a generic m'-region—is what one would need to control to convert mixing under random inputs into a quantitative statement about Collatz orbits.

Open questions. Several questions naturally suggest themselves.

(1) Can one identify a *deterministic* substitute for the random-input model on which Theorem 3.3 would yield bounds on the empirical mean of v_2 along orbits? An effective expander-style mixing inequality for the carry automaton against bit strings produced by Collatz iteration is, to the best of our knowledge, an open problem.

(2) The closed-form $T^{t-1}(n) = (3/2)^{t-1}(n+1) - 1$ extends naturally to a multi-segment epoch decomposition. Quantitative control of the lengths $t_1, t_2, \dots$ of successive segments along an orbit is closely related to the equidistribution of $\{n \log_2 3\}$ on $[0, 1)$ for orbit values n, a question related to but not the same as the analogous problem on $\mathbb{Z}_2$ [4].

(3) The transition matrix M of (4) is doubly stochastic and has the symmetry of the path graph $A - B - C$ with self-loops at the endpoints. Generalizations of $3n+1$ to maps $an+b$ presumably yield similar finite-state carry automata; characterizing when the spectral gap is at least $1/2$ would be of independent interest.

Reproducibility. The eigenvalue computation in Theorem 3.3 can be reproduced in any computer algebra system (or by hand, as we have done). The closed form in Proposition 4.1 can be checked on any odd n in a few lines of code; numerical verification on all odd $n \leq 2^{18}$ shows perfect agreement with the formula.

7 Conclusion

We have presented two elementary, self-contained results about the accelerated Collatz map. The three-state carry automaton (Theorem 3.3) gives a clean finite-state description of the binary computation $n \mapsto 3n+1$ together with a sharp spectral gap of 1/2 under the random-input model. The closed-form shift identity (Proposition 4.1) describes the deterministic trailing-1 epoch as a pure multiplicative scaling $x \mapsto (3/2)^{t-1}x$ in the shifted coordinate $x = n+1$. Neither result purports to resolve the Collatz conjecture; together they form a small piece of the geometric-combinatorial scaffolding that any future quantitative theory of Collatz orbits will likely need to incorporate.

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