

No Finite Local Potential Function Is a Strict Lyapunov Function for the Accelerated Collatz Map: An Impossibility Result

Lightman Chang
Independent Researcher
lightman.chang@gmail.com

Abstract

A standard strategy for attacking the Collatz conjecture is to seek a *Lyapunov function*: a real-valued map on the positive integers that strictly decreases under the accelerated Collatz dynamics $S(n) = (3n+1)/2^{v_2(3n+1)}$. Any such function would force the absence of nontrivial cycles and divergent orbits. A natural family of candidates consists of the *finite local* potential functions of the form $V(n) = \log_2 n + g(n \bmod 2^k)$, where $k \geq 1$ is a fixed precision and $g: \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{R}$ is an arbitrary correction term. We prove (Theorem 3.1) that for every fixed $k \geq 1$ and every choice of g , there exist infinitely many odd integers n on which V does not decrease under S , so V fails to be a strict Lyapunov function for S . The obstruction is structural: the trailing-1 block can have arbitrary length, and a finite-precision correction can compensate at most $O(1)$ growth, while the trailing-1 block of length ℓ produces a multiplicative growth of $(3/2)^\ell$ before any cancellation. Our result rules out an entire class of potential-function attacks on Collatz and complements Conway's [1] undecidability theorems for generalized Collatz problems.

Keywords: Collatz map, $3x+1$ problem, Lyapunov function, impossibility result, 2-adic valuation.

MSC 2020: Primary 11B37; Secondary 37B99, 11A63.

1 Introduction

The Collatz conjecture asserts that for the map $T: \mathbb{N} \rightarrow \mathbb{N}$ defined by $T(n) = n/2$ for n even and $T(n) = 3n+1$ for n odd, every orbit reaches 1. After nearly a century, it remains open; the strongest known density result is due to Tao [5], and the standard surveys are Lagarias [3, 4]. Conway [1] showed that the natural family of generalizations of T is universal (Turing-complete), so any proof of Collatz must exploit specific arithmetic features of the constants $(3, 2, 1)$.

A frequently proposed strategy is to construct a real-valued function $V: \mathbb{N}_{\text{odd}} \rightarrow \mathbb{R}$ that strictly decreases under the accelerated Collatz map

$$S: \mathbb{N}_{\text{odd}} \rightarrow \mathbb{N}_{\text{odd}}, \quad S(n) = \frac{3n+1}{2^{v_2(3n+1)}}. \quad (1)$$

Such a V is called a (strict) *Lyapunov function* for S . The existence of any such V that is bounded below (or merely well-founded) would immediately rule out cycles and divergent orbits, and hence prove the Collatz conjecture.

The natural ansatz. On any orbit, S scales the iterate by approximately $3/2^{v_2}$, so the leading-order behavior of $\log_2 n$ along an orbit is governed by $\log_2 3 - v_2 \approx 1.585 - v_2$. This suggests an ansatz of the form

$$V(n) = \log_2 n + g(n \bmod 2^k), \quad (2)$$

where $k \geq 1$ is a fixed integer (the “locality precision”) and $g: \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{R}$ is a correction term whose role is to absorb the fluctuations of $v_2(3n + 1)$ that depend on the low bits of n . Many variants of this idea are folkloric; small- k versions can be checked by hand and seen to fail.

What we prove. We prove that the failure is structural: for *every* fixed k and *every* choice of g , the function V in (2) fails to be a strict Lyapunov function for S . More precisely, there exist infinitely many odd integers n on which $V(S(n)) \geq V(n)$. The obstruction comes from the trailing-1 blocks: an odd integer of the form $n = 2^\ell - 1$ has a deterministic prefix of $\ell - 1$ “bad” steps, each contributing $\log_2(3/2)$ of growth before any cancellation, while g takes values in a fixed finite set and can contribute at most a fixed constant of cancellation.

What we do not prove. We do not prove (and do not believe) that no Lyapunov function exists at all. The result rules out only the natural finite-precision local ansatz (2). It does not rule out (a) potentials of the form $\log_2 n + g(n \bmod 2^{k(n)})$ with $k(n) \rightarrow \infty$, (b) potentials taking g to be a function of more elaborate features (e.g., $v_2(n + 1)$ itself, or block-structure of the binary expansion), or (c) ordinal-valued potentials. We discuss these extensions in Section 4.

Why the result is useful. Various authors have tried and abandoned the ansatz (2) for small k , often after lengthy ad-hoc case analysis. Our theorem replaces this by a one-sentence obstruction: the family is dead because $\log_2(3/2) > 0$ and trailing-1 blocks are unbounded in length. This frees the search for Lyapunov-style attacks on Collatz to focus on genuinely non-local potentials.

Paper structure. Section 2 fixes notation and establishes the closed-form behavior of S on the trailing-1 block (the only Collatz-specific input we need). Section 3 proves Theorem 3.1. Section 4 discusses extensions and limitations.

2 Preliminaries

We write $\mathbb{N}_{\text{odd}} = \{1, 3, 5, \dots\}$ for the positive odd integers and $v_2(m)$ for the 2-adic valuation of $m \in \mathbb{Z} \setminus \{0\}$. For an odd integer $n \geq 1$ we set

$$t(n) = v_2(n + 1) \geq 1,$$

the number of trailing 1-bits in the binary expansion of n . Thus $n = 2^{t(n)}m' - 1$ with m' odd.

Definition 2.1 (Strict Lyapunov function). A function $V: \mathbb{N}_{\text{odd}} \rightarrow \mathbb{R}$ is a *strict Lyapunov function* for S if $V(S(n)) < V(n)$ for every $n \in \mathbb{N}_{\text{odd}}$ with $n > 1$.

Definition 2.2 (Finite local potential). For a fixed integer $k \geq 1$ and a function $g: \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{R}$, the *finite local potential of precision k with correction g* is

$$V_{k,g}(n) = \log_2 n + g(n \bmod 2^k), \quad n \in \mathbb{N}_{\text{odd}}.$$

We will need only one Collatz-specific fact, recorded here as a lemma; it is a special case of a folkloric closed form (see, e.g., [3, Lemma 2.1]).

Lemma 2.3 (Trailing-1 block). *Let $n \geq 3$ be odd with $t(n) = t \geq 2$. Then for $j = 1, 2, \dots, t - 1$ the iterate $S^j(n)$ is well-defined as an odd integer and*

$$S^j(n) = 2^{t-j} \cdot 3^j \cdot m' - 1,$$

where $n = 2^t m' - 1$. In particular,

$$\log_2 S^j(n) = \log_2 n + j \cdot \log_2(3/2) + \varepsilon_j(n), \tag{3}$$

where $|\varepsilon_j(n)| \leq 2^{-(t-j)-\log_2 m'}$ tends to zero as $n \rightarrow \infty$ along $n = 2^t m' - 1$ with $t - j \geq 1$ and m' fixed.

Proof. Write $n = 2^t m' - 1$. Then $3n + 1 = 3 \cdot 2^t m' - 2 = 2(3 \cdot 2^{t-1} m' - 1)$, and since $3 \cdot 2^{t-1} m' - 1$ is odd (as $2^{t-1} m'$ is even when $t \geq 2$), we have $v_2(3n + 1) = 1$ and $S(n) = 3 \cdot 2^{t-1} m' - 1 = 2^{t-1} \cdot (3m') - 1$. But $3m'$ is odd (product of two odd numbers), so $S(n)$ has the same form $S(n) = 2^{t-1} m'_1 - 1$ with $m'_1 = 3m'$ odd and $t(S(n)) = t - 1$. Iterating, $S^j(n) = 2^{t-j} \cdot 3^j m' - 1$ for $0 \leq j \leq t - 1$, valid because $t - j \geq 1$ keeps $S^j(n)$ in the trailing-1 regime.

For (3),

$$S^j(n) = 2^{t-j} \cdot 3^j m' - 1 = 3^j \cdot \frac{2^t m' - 2^j}{2^j} = \frac{3^j}{2^j} (n + 1) - 1 = \frac{3^j}{2^j} n + \frac{3^j - 2^j}{2^j},$$

so

$$\log_2 S^j(n) = \log_2 n + j \log_2(3/2) + \log_2 \left(1 + \frac{3^j - 2^j}{2^j n} \right).$$

For $n \geq 2^t \cdot 1 - 1$ and $j \leq t - 1$ the third term is $O(2^{-(t-j)})$ uniformly in j , giving the claimed bound on $\varepsilon_j(n)$. \square

Remark 2.4. We use (3) only to control the dominant $j \log_2(3/2)$ term; the error $\varepsilon_j(n)$ will be made arbitrarily small by taking n large. No effort to optimize the constant in the error bound is needed.

3 The Impossibility Theorem

3.1 Statement

Theorem 3.1 (Main result). *Let $k \geq 1$ be a fixed integer. For every function $g: \mathbb{Z}/2^k \mathbb{Z} \rightarrow \mathbb{R}$, the finite local potential $V_{k,g}(n) = \log_2 n + g(n \bmod 2^k)$ fails to be a strict Lyapunov function for the accelerated Collatz map S . More precisely, there exist infinitely many odd integers $n > 1$ such that*

$$V_{k,g}(S(n)) \geq V_{k,g}(n).$$

3.2 Proof

Proof of Theorem 3.1. Fix $k \geq 1$ and an arbitrary function $g: \mathbb{Z}/2^k \mathbb{Z} \rightarrow \mathbb{R}$. Since $\mathbb{Z}/2^k \mathbb{Z}$ is finite, the function g is bounded; set

$$G := \max_{r \in \mathbb{Z}/2^k \mathbb{Z}} g(r) - \min_{r \in \mathbb{Z}/2^k \mathbb{Z}} g(r) \in [0, \infty). \quad (4)$$

Note that G depends on g but is finite (since g takes only 2^k values).

Choose any integer ℓ large enough that

$$\ell \cdot \log_2(3/2) > G + 1. \quad (5)$$

This is possible because $\log_2(3/2) = \log_2 3 - 1 \approx 0.585$ is strictly positive, so the left-hand side grows without bound in ℓ . Concretely, $\ell \geq \lceil (G + 1) / \log_2(3/2) \rceil$ suffices.

For each integer $M \geq 1$ define the odd integer

$$n_M := 2^{\ell+k} \cdot M - 1. \quad (6)$$

Then n_M has the following properties.

(i) n_M is odd, and $t(n_M) = \ell + k \geq \ell$. Indeed, $n_M + 1 = 2^{\ell+k} M$, so $v_2(n_M + 1) = \ell + k$ because M may have its own factors of 2, but we only used $\ell + k$ explicitly; if M is itself even the

trailing-1 count would be larger, which only helps. Restrict to odd M to ensure $t(n_M) = \ell + k$ exactly; we will make this restriction below.

(ii) The residue $n_M \bmod 2^k$ is independent of M : indeed, $n_M \equiv -1 \equiv 2^k - 1 \pmod{2^k}$ for every M , since $2^{\ell+k}M \equiv 0 \pmod{2^k}$. Consequently,

$$g(n_M \bmod 2^k) = g(2^k - 1) \quad \text{is the same constant for every } M.$$

(iii) By Lemma 2.3 applied with $t = \ell + k$ and $j = \ell$, the iterate $S^\ell(n_M)$ is well-defined and satisfies

$$S^\ell(n_M) = 2^k \cdot 3^\ell \cdot M - 1.$$

This number is odd, has $t(S^\ell(n_M)) = k$ trailing 1-bits when M is odd (since $3^\ell M$ is then odd), and most importantly

$$S^\ell(n_M) \equiv 2^k - 1 \equiv -1 \pmod{2^k},$$

so $g(S^\ell(n_M) \bmod 2^k) = g(2^k - 1)$ as well.

Restricting now to odd $M \geq 1$, we have for every such M :

$$\begin{aligned} V_{k,g}(S^\ell(n_M)) - V_{k,g}(n_M) &= [\log_2 S^\ell(n_M) - \log_2 n_M] + [g(S^\ell(n_M) \bmod 2^k) - g(n_M \bmod 2^k)] \\ &= [\log_2 S^\ell(n_M) - \log_2 n_M] + 0, \end{aligned} \quad (7)$$

where the second bracket vanishes by (ii) and (iii).

By Lemma 2.3 (specifically (3)),

$$\log_2 S^\ell(n_M) - \log_2 n_M = \ell \cdot \log_2(3/2) + \varepsilon(M), \quad (8)$$

where $\varepsilon(M) \rightarrow 0$ as $M \rightarrow \infty$ (the error $\varepsilon_\ell(n_M)$ from Lemma 2.3 satisfies $|\varepsilon(M)| \leq C/(2^k M)$ for a constant C depending on ℓ , since $n_M \asymp 2^{\ell+k}M$). Choose M_0 so large that $|\varepsilon(M)| < 1$ for all $M \geq M_0$.

Combining (7), (8) and the choice (5), we obtain that for every odd $M \geq M_0$,

$$V_{k,g}(S^\ell(n_M)) - V_{k,g}(n_M) = \ell \cdot \log_2(3/2) + \varepsilon(M) > G + 1 - 1 = G \geq 0. \quad (9)$$

So far we have shown that the ℓ -step potential difference $V_{k,g}(S^\ell(n_M)) - V_{k,g}(n_M)$ is strictly positive. We now deduce the existence of a single step on which $V_{k,g}$ does not decrease. Telescoping,

$$V_{k,g}(S^\ell(n_M)) - V_{k,g}(n_M) = \sum_{j=0}^{\ell-1} [V_{k,g}(S^{j+1}(n_M)) - V_{k,g}(S^j(n_M))].$$

Since the left-hand side is strictly positive by (9), at least one summand on the right-hand side is strictly positive: there exists an index $j^*(M) \in \{0, 1, \dots, \ell - 1\}$ for which

$$V_{k,g}(S^{j^*(M)+1}(n_M)) > V_{k,g}(S^{j^*(M)}(n_M)).$$

In particular, on the odd integer $\tilde{n}_M := S^{j^*(M)}(n_M)$ the potential strictly fails to decrease.

It remains to verify that the values $\{\tilde{n}_M : M \in \mathbb{N}_{\text{odd}}, M \geq M_0\}$ are infinite (i.e., that we have produced *infinitely many* witnesses). Since $\log_2 \tilde{n}_M \geq \log_2 n_M - O(\ell) \rightarrow \infty$ as $M \rightarrow \infty$ (the bad-step segment of length at most $\ell - 1$ can shrink the iterate by at most a factor of $(2/3)^{\ell-1}$ in log-scale, a finite amount), the set $\{\tilde{n}_M\}$ is unbounded and hence infinite.

This proves the existence of infinitely many odd integers n on which $V_{k,g}(S(n)) \geq V_{k,g}(n)$, as claimed. \square

3.3 Quantitative consequence

Corollary 3.2 (Required precision grows). *Fix any constant $A > 0$ and let $k \geq 1$. For every $g: \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{R}$ with $\max g - \min g \leq A$, the function $V_{k,g}$ admits infinitely many odd integers n on which $V_{k,g}(S(n)) - V_{k,g}(n) \geq 0$. In particular, no uniform bound on the oscillation of g across all k produces a Lyapunov function; the only escape from Theorem 3.1 is to allow g 's oscillation $G(g)$ to grow without bound in k .*

Proof. Direct from (5): any uniform bound $G \leq A$ allows us to choose ℓ once for all k , with the construction of Theorem 3.1 going through as written. \square

4 Discussion

What is ruled out. Theorem 3.1 rules out the entire family of Lyapunov candidates of the form $V_{k,g} = \log_2 n + g(n \bmod 2^k)$ with k fixed and g a function of the residue class. This is, to our knowledge, the most natural class of “elementary” Lyapunov candidates that has been considered in the literature.

What is not ruled out. Our result leaves several variants open.

- (1) *Adaptive precision.* A potential of the form $\log_2 n + g_n(n \bmod 2^{k(n)})$ with $k(n) \rightarrow \infty$ as $n \rightarrow \infty$, and with g_n permitted to vary, is not covered. Corollary 3.2 shows that any such escape requires the oscillation $\max g_n - \min g_n$ to grow without bound in n .
- (2) *Non-residue local features.* Potentials of the form $\log_2 n + h(t(n))$ where $h: \mathbb{N} \rightarrow \mathbb{R}$ depends on the trailing-1 count of n are not directly covered, although a similar argument can be carried out: along the family $n_M = 2^{\ell+k}M - 1$ both $t(n_M)$ and $t(S^\ell(n_M))$ are pinned by construction, so h contributes a fixed amount on each side of (7) and the same lower bound applies. We leave the full statement to the interested reader.
- (3) *Block structure and ordinal potentials.* Goodstein-style ordinal potentials [2] or more elaborate block-structure potentials are not covered. Indeed, the Goodstein function admits a successful ordinal potential (with values in ε_0) showing termination, even though the function is Peano-arithmetic-unprovably-terminating. Whether an analogous transfinite potential exists for Collatz is open.
- (4) *Non-local potentials.* Anything depending on the entire history of an orbit, or on global features of n , escapes our framework.

Relation to Conway’s universality. Conway [1] proved that the natural class of *generalized* Collatz maps (piecewise linear with rational coefficients on residue classes) is computationally universal; in particular, no algorithmic procedure can decide convergence for all such maps. Our Theorem 3.1 is in the same spirit: just as Conway rules out a class of *decision procedures*, we rule out a class of *Lyapunov certificates*. The two results are complementary: Conway shows generalized Collatz is universal; we show that one specific certificate scheme is intrinsically too weak even for the standard $3x + 1$ map.

Relation to existing potential-function attempts. Many authors have explored small- k instances of $V_{k,g}$ and observed failure on specific residue classes (e.g., $n \equiv 7 \pmod{8}$, $n \equiv 15 \pmod{16}$); the present paper consolidates these observations into a uniform statement and a single conceptual obstruction. We do not claim that the impossibility for $k = 2, 3$ is new; we claim that the proof for *all* k is, and that the identification of the obstruction with the trailing-1 blocks clarifies why ad-hoc attempts at low k inevitably failed.

Open problem. What is the smallest growth rate $f(k)$ such that some choice of $g_k: \mathbb{Z}/2^k\mathbb{Z} \rightarrow \mathbb{R}$ with $\max g_k - \min g_k \leq f(k)$ becomes a strict Lyapunov function for S ? Corollary 3.2 shows that bounded f cannot work, but gives no upper bound on the necessary growth. A quantitative statement (e.g., $f(k) = \Omega(\log k)$ or $f(k) = \Omega(k)$) would give a sharper picture of how non-local any successful Lyapunov function must be.

Reproducibility. Theorem 3.1 can be verified numerically by choosing any k and g , then computing $V_{k,g}(S(n_M)) - V_{k,g}(n_M)$ along the trailing-1 family $n_M = 2^{\ell+k}M - 1$ for $M = 1, 3, 5, \dots$ and $\ell = \lceil (G+1)/\log_2(3/2) \rceil$. All arithmetic is exact; no numerical issues arise for $k \leq 20$ and $M \leq 2^{30}$.

5 Conclusion

We have shown that no finite local potential function of the form $\log_2 n + g(n \bmod 2^k)$ with k fixed can serve as a strict Lyapunov function for the accelerated Collatz map S . The obstruction is direct: the trailing-1 block of length ℓ produces growth $\ell \log_2(3/2)$ in $\log_2 n$, while the correction g contributes at most a fixed constant. As a consequence, any successful Lyapunov attack on the Collatz conjecture must employ either adaptive precision ($k = k(n) \rightarrow \infty$), non-residue features (e.g., trailing-1 counts), or genuinely non-local information (e.g., orbit history). Our result delineates the boundary of what the most natural local ansatz can achieve, leaving open whether any of the more elaborate variants succeeds.

References

- [1] J. H. Conway, *Unpredictable iterations*, in: *Proc. Number Theory Conf. (Boulder, 1972)*, pp. 49–52, Univ. Colorado, Boulder, CO, 1972.
- [2] L. Kirby and J. Paris, *Accessible independence results for Peano arithmetic*, *Bull. London Math. Soc.* **14** (1982), 285–293.
- [3] J. C. Lagarias, *The $3x+1$ problem and its generalizations*, *Amer. Math. Monthly* **92** (1985), 3–23.
- [4] J. C. Lagarias (ed.), *The Ultimate Challenge: The $3x+1$ Problem*, American Mathematical Society, Providence, RI, 2010.
- [5] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, *Forum Math. Pi* **10** (2022), e12.