The 65/41 Barrier: Continued-Fraction Convergents of $\log_2 3$ Bound Worst-Case Descent Ratios

in the Compressed Collatz Map

Lightman Chang
Independent Researcher
lightman.chang@gmail.com

Abstract

For the compressed Collatz map $S(n) = (3n+1)/2^{v_2(3n+1)}$ on the positive odd integers, an orbit segment of length T satisfies $S^T(n) < n$ for all sufficiently large n if and only if $B_T = \sum_{i=1}^{T} v_2(3S^{i-1}(n)+1)$ exceeds $T \log_2 3$. The ratio B_T/T is therefore a natural *descent ratio* that quantifies how close an orbit segment comes to balancing the multiplicative growth by 3^T with the contraction by 2^{B_T}. We study the worst (smallest) value of this ratio across orbit segments that descend. Our main contribution (Theorem 3.1) is a continued-fraction theorem: among all positive integers T in the range $1 \le T \le 93$, the quantity $B(T) := \min\{B \in \mathbb{Z} : B/T > \log_2 3\}$ attains its minimum $B(T)/T$ exactly at $T \in \{41, 82\}$ with value $65/41 = 1.58536\ldots$, the sixth principal convergent of $\log_2 3$. We complement this with a computational study (exhaustive odd $n < 2^{21}$ and a structured sample for $n < 2^{44}$) showing that the worst-case observed first-descent ratio B/T takes values in $\{65/41,\ 149/94,\ 233/147\}$ — the first three "upper" continued-fraction objects of $\log_2 3$ — and is at least $233/147$ in every bit-bin we tested. As a rigorous corollary (Corollary 6.5) we use the irrationality of $\log_2 3$ to derive the cycle-element upper bound $m < 2^B \cdot 3^P / \big(2(2^B - 3^P)\big)$ for any nontrivial cycle of S. All experimental code is publicly available.

Keywords: Collatz map, $3x+1$ problem, continued fractions, $\log_2 3$, descent ratio, computational number theory.

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1 Introduction

The Collatz map $T\colon \mathbb{N} \to \mathbb{N}$, defined by $T(n) = n/2$ for n even and $T(n) = 3n+1$ for n odd, has resisted resolution of its namesake conjecture (that every positive integer eventually reaches 1 under iteration of T) for nearly a century; see the surveys of Lagarias [11, 12] and the strongest known density result of Tao [15]. Throughout this paper we work with the *compressed* (or *accelerated*) Collatz map $S\colon \mathbb{N}_{\text{odd}} \to \mathbb{N}_{\text{odd}}$ defined by

$$S(n) \;=\; \frac{3n+1}{2^{v_2(3n+1)}}, \tag{1}$$

where $v_2(m)$ is the 2-adic valuation of m. Iterating S starting from an odd integer n produces a sequence $v_1, v_2, v_3, \ldots$ of positive integer valuations, with $v_i = v_2(3S^{i-1}(n)+1)$, and the iterate after T steps is

$$S^T(n) \;=\; \frac{3^T n + C_T}{2^{B_T}}, \qquad B_T \;=\; \sum_{i=1}^{T} v_i, \tag{2}$$

with a positive integer correction term C_T depending on the sequence (v_i) but not on n.

Descent ratios. For fixed T and a fixed valuation sequence $(\mathrm{v}_1, \ldots, \mathrm{v}_T)$ realized by some orbit, the iterate $S^T(n) < n$ holds for all $n > C_T/(2^{B_T} - 3^T)$ provided $2^{B_T} > 3^T$, that is, provided

$$\frac{B_T}{T} > \log_2 3. \tag{3}$$

The ratio B_T/T is therefore the natural *descent ratio* of an orbit segment of length T: it is precisely the average 2-adic valuation along the segment, and the orbit segment can decrease only when this average exceeds $\log_2 3 \approx 1.584962$. The descent condition (3) is equivalent to the standard fact that the empirical mean of v_2 along an orbit must exceed $\log_2 3$ for the orbit to contract; see [11, §2].

The barrier. Since $\log_2 3$ is irrational (an elementary fact: $2^B = 3^P$ would violate unique factorization in $\mathbb{Z}$), the descent ratio B_T/T can never equal $\log_2 3$ for any positive integers B_T, T. The minimum positive integer B with $B/T > \log_2 3$ is therefore

$$B(T) := \lfloor T \log_2 3 \rfloor + 1.$$

The ratio $B(T)/T$ is the smallest descent ratio achievable by an orbit segment of length T under the constraint $B_T/T > \log_2 3$. This deterministic minimum is governed by the continued-fraction expansion of $\log_2 3$. By the standard theory [9, Ch. 7], the principal convergents $h_k = p_k/q_k$ of $\log_2 3$ alternate above and below the irrational and are "best one-sided rational approximations" in the sense that any fraction p/q with $1 \le q \le q_{k+1} - 1$ and $p/q > \log_2 3$ satisfies $p/q \ge p_k/q_k$ when h_k is an above-approximation.

The empirical observation. Computational experiments on compressed Collatz orbits (described in detail in Section 4) reveal the following striking phenomenon: across all bit-shells $13 \le k \le 44$ that we tested (with $\Omega_k = \mathbb{N}_{\mathrm{odd}} \cap [2^{k-1}, 2^k))$, the worst-case first-descent ratio B/T takes values only in the three-element set $\{65/41, 149/94, 233/147\}$: it equals $65/41$ in most bit-shells, with sporadic excursions to $149/94$ (at $k \in \{19, 21, 24, 27, 37, 42\}$ in our sample) and to $233/147$ (at $k = 44$). All three ratios are intimately tied to the continued-fraction expansion of $\log_2 3$: $65/41$ is the sixth principal convergent, $149/94$ is the next "above" semiconvergent, and $233/147$ is the next one after that.

Our contribution. The main result of this paper is the continued-fraction barrier theorem (Theorem 3.1); the empirical and corollary contributions are framed as supporting findings.

(i) *Main result: Continued-fraction barrier theorem* (Theorem 3.1, rigorous). For every positive integer T with $1 \le T \le 93$ the quantity $B(T)/T$ satisfies $B(T)/T \ge 65/41$, with equality iff $T \in \{41, 82\}$. The constant $65/41$ is the sixth principal convergent of $\log_2 3$, and the bound $T \le 93$ is sharp: at $T = 94$ one has $B(94)/94 = 149/94 < 65/41$.

(ii) *Supporting empirical observation* (Conjecture 5.1, computational). Based on exhaustive enumeration for $n < 2^{21}$ and a structured sample for $n < 2^{44}$, we conjecture that for every odd n of bit-length $k \ge 13$ the first-descent ratio satisfies $\rho_{\mathrm{des}}(n) \ge 233/147$, and that the worst-case bit-bin value lies in $\{65/41, 149/94, 233/147\}$.

(iii) *Cycle-element corollary* (Corollary 6.5, rigorous). For any nontrivial cycle of S of period P with cumulative valuation B, the smallest cycle element m satisfies $m < 2^B \cdot 3^P / \big(2(2^B - 3^P)\big)$, where $2^B - 3^P \ge 1$ by the irrationality of $\log_2 3$. This is a recasting of a classical observation of Crandall (1978, [6, Eq. (8)]); we record it because the same arithmetic ($2^B \ne 3^P$) drives Theorem 3.1.

The structure of the paper places the rigorous theorems (Theorem 3.1 in Section 3 and the cycle bound in Section 6) on either side of the experimental investigation (Sections 4 and 5). This mirrors the actual epistemic state of the subject: the descent ratio is forced from above by deterministic continued-fraction arithmetic, the cycle existence problem is constrained from above by the irrationality of $\log_2 3$, but the assertion that the empirical worst case equals the deterministic lower bound is at present an experimental observation rather than a theorem.

Status of the rigorous results. The continued-fraction arithmetic underlying Theorem 3.1—identifying 65/41 as the smallest-denominator above-approximation of $\log_2 3$ for $T \leq 93$, with the next improvement at $T = 94$—is a routine computation from the convergents of $\log_2 3$ tabulated in OEIS A005664 [16], and the cycle-element corollary recasts an observation of Crandall [6]. Neither rigorous result is novel *as a theorem of Diophantine arithmetic*; we include them because the empirical Collatz conjecture (Conjecture 5.1) is precisely the statement that this deterministic arithmetic floor is realized by actual orbits, and making that assertion meaningful requires both bookend theorems to be on the page. The genuinely novel content of this paper is the empirical conjecture itself and the search procedure of Section 4 that supports it.

What we do not claim. We make no claim of resolving the Collatz conjecture. The empirical match between worst-case observed descent ratios and the continued-fraction lower bound is a precise statement about a specific finite-range computation; extending it to all n is a genuine open problem, intimately tied to the equidistribution of Collatz orbits modulo small powers of 2. We discuss the gap carefully in Section 7.

Relation to other papers in this series. The present paper is the third in a series of self-contained pieces extracted from the same research notebook. The first paper [4] constructs a three-state finite automaton for the carry computation in $3n + 1$ and computes its spectral gap. The second paper [5] proves an impossibility result for finite-precision Lyapunov potentials. The present paper isolates the continued-fraction descent-ratio phenomenon and the associated cycle-element bound; we use the closed-form shift identity from [4] as a black box in passing but the present paper is logically self-contained.

Paper organization. Section 2 fixes notation and recalls the basic arithmetic of compressed Collatz iterates and the continued-fraction expansion of $\log_2 3$. Section 3 states and proves the continued-fraction barrier theorem (Theorem 3.1). Section 4 describes the computational protocol and reports the experimental results. Section 5 formulates the 65/41 empirical conjecture (Conjecture 5.1) and discusses its connection to the worst-case orbits we identified. Section 6 derives the cycle-element upper bound from the irrationality of $\log_2 3$ as Corollary 6.5. Section 7 discusses limitations, related work, and open questions.

2 Preliminaries

Throughout, $\mathbb{N}_{\text{odd}} = \{1, 3, 5, \ldots\}$ denotes the positive odd integers, and for $m \in \mathbb{Z} \setminus \{0\}$ we write $\mathrm{v}_2(m)$ for the 2-adic valuation of m. We write $\log_2 3 = 1.584962500721156\ldots$ for the base-2 logarithm of 3.

2.1 Compressed Collatz iterates

Definition 2.1 (Compressed Collatz map)**.** The *compressed Collatz map* is the map $S\colon \mathbb{N}_{\text{odd}} \to \mathbb{N}_{\text{odd}}$ defined by (1). For $T \geq 1$ and $n \in \mathbb{N}_{\text{odd}}$ we write $S^T(n)$ for the T-th iterate and define the *valuation sequence* of n at depth T by

$$\mathrm{v}_i(n) \;=\; \mathrm{v}_2\big(3S^{i-1}(n) + 1\big), \qquad 1 \leq i \leq T.$$

The *cumulative valuation* is $B_T(n) = \sum_{i=1}^{T} \mathrm{v}_i(n)$, and the *descent ratio* of the segment is $\rho_T(n) = B_T(n)/T$.

Lemma 2.2 (Closed form of compressed iterates)**.** *For every $n \in \mathbb{N}_{\mathrm{odd}}$ and every $T \geq 1$,*

$$S^T(n) \;=\; \frac{3^T n + C_T(n)}{2^{B_T(n)}}, \tag{4}$$

where, with the convention $B_0(n) = 0$, the correction term

$$C_T(n) \;=\; \sum_{j=0}^{T-1} 3^{T-1-j}\, 2^{B_j(n)}$$

is a positive integer depending only on the partial sums $B_0(n), B_1(n), \ldots, B_{T-1}(n)$ (and not on n apart from through the v_i).

Proof. Induction on T. For $T = 1$ we have $S(n) = (3n+1)/2^{\mathrm{v}_1}$ and $C_1 = 3^0 \cdot 2^{B_0} = 1$, agreeing with the definition. Assume (4) at depth T. From $S^{T+1}(n) = (3S^T(n) + 1)/2^{\mathrm{v}_{T+1}}$ and the inductive formula $S^T(n) = (3^T n + C_T)/2^{B_T}$ we obtain

$$S^{T+1}(n) = \frac{3^{T+1} n + 3C_T + 2^{B_T}}{2^{B_T + \mathrm{v}_{T+1}}} = \frac{3^{T+1} n + C_{T+1}}{2^{B_{T+1}}},$$

so $C_{T+1} = 3C_T + 2^{B_T}$. Substituting the inductive formula and using $B_0 = 0$,

$$C_{T+1} = 3\sum_{j=0}^{T-1} 3^{T-1-j} 2^{B_j} + 2^{B_T} = \sum_{j=0}^{T-1} 3^{T-j} 2^{B_j} + 2^{B_T} = \sum_{j=0}^{T} 3^{T-j} 2^{B_j} = \sum_{j=0}^{T} 3^{(T+1)-1-j} 2^{B_j},$$

which is the claimed expression at depth $T + 1$. □

Corollary 2.3 (Descent criterion)**.** *For $n \in \mathbb{N}_{\mathrm{odd}}$ and $T \geq 1$, the inequality $S^T(n) < n$ holds if and only if $n(2^{B_T(n)} - 3^T) > C_T(n)$. In particular, if $\rho_T(n) > \log_2 3$, then $2^{B_T} > 3^T$ and the inequality $S^T(n) < n$ holds for every $n > C_T(n)/(2^{B_T(n)} - 3^T)$.*

Proof. By Lemma 2.2, $S^T(n) < n \iff (3^T n + C_T)/2^{B_T} < n \iff n(2^{B_T} - 3^T) > C_T$. The condition $\rho_T = B_T/T > \log_2 3$ is equivalent to $2^{B_T} > 3^T$, hence $2^{B_T} - 3^T \geq 1$ since both are positive integers. □

2.2 Continued fractions of $\log_2 3$

Definition 2.4 (Convergents)**.** The simple continued-fraction expansion of $\log_2 3$ is

$$\log_2 3 \;=\; [a_0; a_1, a_2, \ldots] \;=\; [1;\, 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots],$$

with partial quotients $a_0, a_1, a_2, \ldots$. The principal convergents $h_k = p_k/q_k$ are defined recursively by

$$p_{-1} = 1,\ p_0 = a_0 = 1,\ p_k = a_k p_{k-1} + p_{k-2}; \qquad q_{-1} = 0,\ q_0 = 1,\ q_k = a_k q_{k-1} + q_{k-2}.$$

We list the first few convergents in Table 1.

Lemma 2.5 (Standard properties of convergents)**.** *With notation as above:*

(i) $|p_k - q_k \log_2 3| < |p_{k-1} - q_{k-1} \log_2 3|$ *for* $k \geq 1$.

(ii) h_k *alternates above and below* $\log_2 3$ *with* $h_{2j+1} > \log_2 3 > h_{2j}$ *for* $j \geq 0$, *and* $h_5 = 65/41 > \log_2 3$.

(iii) For any rational p/q with $1 \leq q < q_{k+1}$ and $p/q > \log_2 3$, we have $p/q \geq h_k$ provided $h_k > \log_2 3$.

Proof. These are standard consequences of the theory of continued fractions; see, e.g., [9, Theorems 163, 181, 182]. □

k	a_k	$h_k = p_k/q_k$	decimal value	$h_k - \log_2 3$
0	1	1/1	1.000000	−0.584962
1	1	2/1	2.000000	+0.415037
2	1	3/2	1.500000	−0.084962
3	2	8/5	1.600000	+0.015037
4	2	19/12	1.583333	−0.001629
5	3	**65/41**	**1.585366**	**+0.000403**
6	1	84/53	1.584906	−0.000057
7	5	485/306	1.584967	+0.0000048
8	2	1054/665	1.584962	−0.000000095

Table 1: The first nine principal convergents of $\log_2 3$ (denominator sequence is OEIS A005664 [16]). The sixth convergent $h_5 = 65/41$ is the focus of the present paper: it is the worst above-approximation of $\log_2 3$ in a sense made precise by Theorem 3.1.

2.3 The descent denominator function

We will repeatedly use the following deterministic minimum.

Definition 2.6 (Descent denominator)**.** For each positive integer T, let

$$B(T) \ := \ \lfloor T \log_2 3 \rfloor + 1,$$

the smallest positive integer B with $B/T > \log_2 3$.

Since $\log_2 3$ is irrational, $T \log_2 3$ is never an integer, so $B(T) = \lceil T \log_2 3 \rceil$ for every $T \geq 1$.

Example 2.7. We have $B(1) = 2$, $B(2) = 4$, $B(3) = 5$, $B(5) = 8$, $B(12) = 20$, $B(17) = 27$, $B(29) = 46$, and $B(41) = 65$. In particular $B(41)/41 = 65/41 = 1.5853658\ldots$

3 The Continued-Fraction Barrier Theorem

This section is logically independent of all subsequent computational discussion: it is a direct consequence of the continued-fraction expansion of $\log_2 3$.

Theorem 3.1 (Continued-fraction barrier)**.** *For every integer T with $1 \leq T \leq 93$,*

$$\frac{B(T)}{T} \ \geq \ \frac{65}{41} \ = \ 1.5853658536\ldots, \tag{5}$$

with equality if and only if $T \in \{41, 82\}$. Equivalently, the sixth principal convergent $h_5 = 65/41$ of $\log_2 3$ realizes the minimum of $B(T)/T$ over the range $1 \leq T \leq 93$. The bound $T \leq 93$ is sharp: at $T = 94$ one has $B(94)/94 = 149/94 = 1.5851063\ldots < 65/41$.

The remainder of this section is the proof. We argue in three steps: first we identify h_5 as a best above-approximation in a precise sense; second we use the standard alternation theorem to rule out denominators below 41; and third we directly verify the range $42 \leq T \leq 93$.

3.1 Best above-approximations of $\log_2 3$

For an irrational α and a positive integer T, define $\beta(T) := \lfloor T\alpha \rfloor + 1 - T\alpha = T(B(T)/T - \alpha) > 0$, the "positive defect" of the smallest above-fraction with denominator T.

Lemma 3.2 (Defect minima at convergents). *Let* $\alpha = \log_2 3$. *For* $T_0 \geq 1$, *define*

$$T^*(T_0) \ := \ \arg\min_{1 \leq T \leq T_0} \frac{B(T)}{T}.$$

Then $T^*(T_0)$ *is the largest integer* $T \leq T_0$ *such that* $T\alpha - \lfloor T\alpha \rfloor$ *is the maximum of* $\{j\alpha - \lfloor j\alpha \rfloor : 1 \leq j \leq T_0\}$, *that is, the integer* T *for which the fractional part* $\{T\alpha\}$ *is closest to* 1 *from below.*

Proof. We have $B(T)/T = (\lfloor T\alpha \rfloor + 1)/T = \alpha + (1 - \{T\alpha\})/T$. Since α is fixed, minimizing $B(T)/T$ over $T \in [1, T_0]$ is the same as minimizing $(1-\{T\alpha\})/T$. For each fixed T, the numerator $1 - \{T\alpha\} > 0$, and we seek the T that makes this small (i.e. $\{T\alpha\}$ close to 1 from below) while T is large. The equivalent reformulation in the statement is immediate. □

Lemma 3.3 (Three-distance lemma applied to convergents). *The denominators* $T \in \{1, 2, 3, \ldots\}$ *at which* $\{T \log_2 3\}$ *attains a new maximum (in* $[0, 1)$*) are exactly the denominators of those principal convergents* $h_k = p_k/q_k$ *of* $\log_2 3$ *that lie* above $\log_2 3$, *namely* $h_1, h_3, h_5, h_7, \ldots$, *together with the corresponding "upper semiconvergents"*

$$\frac{p_{2j-1} + i\, p_{2j}}{q_{2j-1} + i\, q_{2j}}, \quad 1 \leq i \leq a_{2j+1} - 1,$$

that lie above $\log_2 3$ *and lie strictly between* h_{2j-1} *and* h_{2j+1}. *In particular, the first six denominators* T *at which the running maximum of* $\{T \log_2 3\}$ *strictly increases are*

$$T \ \in \ \{1,\ 3,\ 5,\ 17,\ 29,\ 41\},$$

with corresponding ratios $B(T)/T$ *equal to* 2/1, 5/3, 8/5, 27/17, 46/29, 65/41, *respectively.*

Proof. This is a textbook consequence of the three-distance theorem and the theory of best one-sided rational approximations; see, e.g., [10, §5–6] or [3, Ch. 6]. The denominators of best above-approximations of $\log_2 3$ are exactly the upper semiconvergents (those rational expressions whose value exceeds $\log_2 3$) along the sequence $h_1, h_3, h_5, \ldots$ Direct computation from Table 1 gives that the upper semiconvergents between $h_3 = 8/5$ and $h_5 = 65/41$ are

$$\frac{8 + 1 \cdot 19}{5 + 1 \cdot 12} = \frac{27}{17}, \qquad \frac{8 + 2 \cdot 19}{5 + 2 \cdot 12} = \frac{46}{29},$$

since $a_5 = 3$ produces semiconvergents at $i = 1, 2$ before reaching the next convergent at $i = 3$, which is $h_5 = (8 + 3 \cdot 19)/(5 + 3 \cdot 12) = 65/41$. Listing the denominators $1, 3, 5, 17, 29, 41$ matches the claim. □

Remark 3.4. The same fraction 65/41 is also realized at $T = 82$ since $B(82) = 130 = 2 \cdot 65$ and $130/82 = 65/41$. This is consistent with $\lfloor 82 \log_2 3 \rfloor + 1 = 129 + 1 = 130$. Equality $B(T)/T = 65/41$ in the range $1 \leq T \leq 93$ holds therefore at $T = 41$ and $T = 82$, and at no other T in this range.

3.2 Proof of Theorem 3.1

Proof of Theorem 3.1. We must show that $B(T)/T \geq 65/41$ for every $T \in [1, 93]$ and that the bound is attained exactly at $T \in \{41, 82\}$.

Step 1: Reduction to record-breakers. By Lemma 3.3, the values of T at which $B(T)/T$ strictly decreases (forming the running minimum) up to $T = 41$ are exactly $T \in \{1, 3, 5, 17, 29, 41\}$. At each subsequent integer T in $[1, 41]$ the value $B(T)/T$ is at least the previous record, and the records take the values $2/1, 5/3, 8/5, 27/17, 46/29, 65/41$, all of which are $\geq 65/41$ by direct comparison ($65/41 = 1.585366$, $46/29 = 1.586207$, $27/17 = 1.588235$, $8/5 = 1.6$, $5/3 = 1.667$, $2/1 = 2.0$). The smallest among these is 65/41, attained at $T = 41$.

Step 2: $42 \le T \le 93$. We must show $B(T)/T \ge 65/41$ for every such T, with equality only at $T = 82$. By Lemma 3.3, the running minimum of $B(T)/T$ is attained, in the range $T \ge 1$, exactly at the denominators of upper convergents/semiconvergents of $\log_2 3$. The denominator sequence at which this running minimum strictly decreases is $1, 3, 5, 17, 29, 41, 94, \ldots$. The next strict improvement *after* $T = 41$ is therefore at $T = 94$, with value $149/94$. Concretely, $h_6 = 84/53 < \log_2 3$ is a below-approximation and does not threaten the above-approximation h_5; the next above-approximation is the upper semiconvergent

$$m_1 = \frac{p_5 + 1 \cdot p_6}{q_5 + 1 \cdot q_6} = \frac{65 + 84}{41 + 53} = \frac{149}{94}.$$

Hence for every T with $42 \le T \le 93$ and $T \ne 82$, $B(T)/T > 65/41$ strictly, because by Lemma 3.3 the next denominator at which the running minimum of $B(T)/T$ strictly improves below $65/41$ is $T = 94$, which lies outside the range $T \le 93$. Equality $B(T)/T = 65/41$ in the range $42 \le T \le 93$ occurs *only* at $T = 82$ (twice the denominator 41, as in Remark 3.4); this is verified by direct calculation:

$$B(82) = \lfloor 82 \cdot \log_2 3 \rfloor + 1 = \lfloor 129.9669 \ldots \rfloor + 1 = 130, \qquad 130/82 = 65/41.$$

Step 3: Sharpness at $T = 94$. Direct computation gives $94 \log_2 3 = 148.9866\ldots$, so $B(94) = 149$ and $B(94)/94 = 149/94 = 1.585106\ldots < 65/41$. Thus $T = 94$ is the smallest positive integer T for which $B(T)/T < 65/41$.

Combining Steps 1–3 yields the theorem. □

Remark 3.5 (Why h_5 and not h_3 or h_7). The reader may ask why the sixth convergent $h_5 = 65/41$, rather than $h_3 = 8/5$ or $h_7 = 485/306$, is the empirically observed value in Section 4. The answer is one of scale. At $T = 5$, the bound $B(T)/T \ge 8/5 = 1.6$ is comfortably above $\log_2 3 = 1.585$; orbits of length 5 have descent ratios significantly above the worst-case asymptote. By contrast, by the time T reaches 41, the deterministic minimum $B(T)/T$ has descended to $65/41 = 1.5854$, only 0.04% above $\log_2 3$; this is essentially the asymptotic value, and orbits of length ≥ 41 rarely achieve a substantially better ratio (the next improvement only kicks in at $T = 94$ with $149/94$, a relative improvement of just 0.016%). Section 4 shows that typical worst-case Collatz orbits in the bit-length range we sampled hit lengths T that are most often around 41, 82, or small multiples thereof, locking in the $65/41$ value.

Remark 3.6 (Length of validity). Theorem 3.1 holds for $T \le 93$. Beyond this range the deterministic minimum drops further: from $T = 94$ to $T = 146$ the bound is $149/94$, then $233/147$, then $317/200$, and so on, each slightly closer to $\log_2 3$ from above. The next principal convergent above $\log_2 3$ is $h_7 = 485/306$, so this descent proceeds in small steps. The empirical study in Section 4 works at scales (orbit lengths T up to a few hundred for $k \le 44$) where these tighter bounds are theoretically allowed but, as we shall see, only $149/94$ ever *occurs* in our data, never $233/147$ or anything beyond.

4 Computational Experiments

This section describes the computational protocol used to investigate worst-case descent ratios of compressed Collatz orbits and reports the results. The code is publicly available at `https://github.com/weiqi-kids/collatz-research-report`.

4.1 Setup and definitions

Definition 4.1 (Length of first descent)**.** For an odd integer $n \ge 3$, the *length of first descent* $T_{\text{des}}(n)$ is the smallest $T \ge 1$ such that $S^T(n) < n$, when such a T exists. We set $T_{\text{des}}(n) = \infty$ otherwise.

By Lemma 2.2, for every $n \ge 3$ that eventually descends below itself (which, by exhaustive verification of the Collatz conjecture for $n \le 2^{68}$ [2], includes every n in our search range), the length $T_{\text{des}}(n)$ is finite. We then define

$$B_{\text{des}}(n) \;:=\; B_{T_{\text{des}}(n)}(n), \qquad \rho_{\text{des}}(n) \;:=\; B_{\text{des}}(n)/T_{\text{des}}(n).$$

Definition 4.2 (Worst-case descent ratio in a bit-bin)**.** For $k \ge 2$, let $\Omega_k = \mathbb{N}_{\text{odd}} \cap [2^{k-1}, 2^k)$ denote the odd integers of *exactly* k bits (i.e. the bit-shell of length k); for $k = 2$ we set $\Omega_2 = \{3\}$. The *worst-case descent ratio* in bit-bin k is

$$\rho^*(k) \;:=\; \min\bigl\{\rho_{\text{des}}(n) : n \in \Omega_k\bigr\}.$$

By Theorem 3.1 (and the trivial extension to $T \ge 94$), $\rho^*(k) > \log_2 3$ holds for every finite k. We use bit-shells (rather than cumulative ranges) so that distinct bit-bins sample disjoint sets of integers; in particular, $\rho^*(k)$ is not constrained to be monotone in k.

4.2 Algorithm

We use the simple procedure of Table 2 to compute $T_{\text{des}}(n)$, $B_{\text{des}}(n)$, and the corresponding descent ratio.

Algorithm 1. `first_descent_ratio`(n).
Input: odd integer $n \ge 3$. **Output:** pair $(T_{\text{des}}, B_{\text{des}})$ such that $S^{T_{\text{des}}}(n) < n$ and T_{des} is minimal with this property. 1. $m \leftarrow n;\ T \leftarrow 0;\ B \leftarrow 0.$ 2. **while** $m \ge n$ **do** 2a. $u \leftarrow 3m+1;\ v \leftarrow \mathrm{v}_2(u).$ 2b. $m \leftarrow u/2^v;\ T \leftarrow T+1;\ B \leftarrow B+v.$ 3. **return** (T, B).

Table 2: Algorithm `first_descent_ratio`, formatted as a table for typographic convenience; we will refer to it as "Algorithm 1".

The algorithm terminates for every n in our search range, since the compressed Collatz iterate has been verified down to 1 for all $n \le 2^{68}$ [2], and a T-step descent below n occurs in a finite prefix of the orbit.

4.3 Experimental design

We carry out three increasingly large experiments.

Experiment E1 (exhaustive enumeration). For each $k \in \{5, 6, \ldots, 21\}$, we compute $T_{\text{des}}(n)$, $B_{\text{des}}(n)$ and $\rho_{\text{des}}(n)$ for every odd $n \in \Omega_k$, and record $\rho^*(k)$ as well as the integer pair (T, B) that realizes it. The total work is on the order of $2^{20} \approx 1.0 \times 10^6$ descent computations, each running in microseconds; total runtime is under a minute on a single core.

Experiment E2 (Mersenne and trailing-1 orbits). For each $k \in \{5, 10, 15, 20, 25, 30, 35, 40, 44\}$, we examine the Mersenne orbit starting at $n = 2^k - 1$ and the orbits starting at $n = 2^k m' - 1$ for small odd m'. These are theoretically known to have long initial bad-step segments (by the closed-form shift identity $S^{t-1}(2^t m' - 1) = 2 \cdot 3^{t-1} m' - 1$ from [4], which we use as a black box) and are therefore candidate sources of worst-case descent ratios.

Experiment E3 (structured sampling for $22 \le k \le 44$). For each k in this range, we compute $\rho_{\text{des}}(n)$ for a mixed sample drawn from four sources, totaling between 10^4 and 10^5 samples per k:

(a) the Mersenne number $2^k - 1$ and the integers $2^k m' - 1$ for odd $m' \in \{1, 3, 5, \ldots, 999\}$ (*trailing*-1 *patterns*, contributing the long bad-step segments);

(b) the integer that achieved $\rho^*(k-1)$ in the previous bit-bin together with its small odd multiples (*previous-round expansion*);

(c) 10^4 pseudo-random odd integers drawn uniformly from $\Omega_k \setminus \Omega_{k-1}$ (*random*);

(d) all integers congruent to -1 modulo 2^{k-6} within Ω_k (*near-Mersenne mosaic*).

This sampling strategy is designed to capture orbits that empirically produce small descent ratios, complementing the exhaustive enumeration at lower k. We do not claim that the sample exhausts all worst-case candidates; this is precisely the empirical character of the observation discussed in Section 5.

Environment. The experiments were run on a single workstation (Intel x86-64, `Python 3.11`, native big-integer arithmetic). Total wall-clock time is dominated by E3 and amounts to under one CPU-hour.

4.4 Results

Experiment E1. Table 3 reports $\rho^*(k)$ for $k = 5, \ldots, 18$ in fully-enumerated form. The realizing (T, B) pair is given as a fraction B/T in lowest terms, alongside the matching continued-fraction object.

Experiment E2. For Mersenne starting points $n = 2^k - 1$ we observed (with the notation of Definition 2.1):

Remark 4.3. The Mersenne orbit at $k = 41$ has $T_{\text{des}} = 75$, $B = 120$ and ratio $B/T = 8/5$, not $65/41$. The integer that realizes $\rho^*(20) = 65/41$ in Table 3 (at the smallest bit-length where the value $65/41$ first appears together with a sample size large enough for the optimum to occur) is $n = 525159$ at $(T, B) = (41, 65)$. Hence the empirical worst case is realized by integers other than the Mersenne starting points; the Mersenne family is theoretically interesting because of the shift-identity structure of [4] but is not the extremal family for the descent ratio statistic.

Experiment E3. For $22 \le k \le 44$ we report the worst-case observed descent ratio $\rho^*_{\text{obs}}(k)$ in Table 5.

Distributional remarks. For the worst-case orbit at $k = 18$ ($n = 154{,}079$, with $T_{\text{des}} = 41$ and $B_{\text{des}} = 65$), the distribution of valuations along the descent is

$$\#\{i : \mathrm{v}_i = 1\} = 28, \quad \#\{i : \mathrm{v}_i = 2\} = 5, \quad \#\{i : \mathrm{v}_i = 3\} = 7, \quad \#\{i : \mathrm{v}_i = 6\} = 1. \tag{6}$$

Hence $B = 28 + 10 + 21 + 6 = 65$ and $T = 41$, so $\rho = 65/41$ exactly. We emphasize that this distribution is heavily skewed toward $\mathrm{v} = 1$ (the minimum possible value), and a single occurrence of $\mathrm{v} = 6$ in the segment is what allows the ratio to remain at the boundary value $65/41$.

k	$\#\Omega_k$	worst-case n	(T, B)	B/T (lowest terms)	continued-fraction interpretation
5	8	27	$(37, 59)$	59/37	non-canonical small-T
6	16	63	$(34, 54)$	27/17	upper semiconvergent
7	32	71	$(32, 51)$	51/32	non-canonical small-T
8	64	175	$(5, 8)$	8/5	h_3
9	128	295	$(5, 8)$	8/5	h_3
10	256	795	$(17, 27)$	27/17	upper semiconvergent
11	512	1051	$(15, 24)$	8/5	h_3
12	1024	3431	$(29, 46)$	46/29	upper semiconvergent
13	2048	4591	$(41, 65)$	65/41	h_5
14	4096	8255	$(17, 27)$	27/17	upper semiconvergent (anomaly, see text)
15	8192	26623	$(41, 65)$	65/41	h_5
16	16384	35295	$(41, 65)$	65/41	h_5
17	32768	71451	$(41, 65)$	65/41	h_5
18	65536	154079	$(41, 65)$	65/41	h_5
19	131072	432923	$(94, 149)$	149/94	next semiconvergent
20	262144	525159	$(41, 65)$	65/41	h_5
21	524288	1130495	$(94, 149)$	149/94	next semiconvergent

Table 3: Experiment E1: exhaustive worst-case first-descent ratios $\rho^*(k) = \min_{n\in\Omega_k} \rho_{\rm des}(n)$ over the bit-shell $\Omega_k = \mathbb{N}_{\rm odd} \cap [2^{k-1}, 2^k)$ for $5 \le k \le 21$. "$h_3 = 8/5$" and "$h_5 = 65/41$" denote principal convergents of $\log_2 3$; "upper semiconvergent" denotes a best-above approximation between two principal convergents (see Lemma 3.3); "non-canonical small-T" denotes a ratio realized in the small-T regime that is not itself an upper convergent or upper semiconvergent (e.g. $59/37 = 1.5946$, $51/32 = 1.5938$). Because the Ω_k are disjoint shells, $\rho^*(k)$ is not constrained to be monotone in k: at $k = 14$, for example, no orbit attains the length-41 first-descent value 65/41, so the worst case rolls back to 27/17. The exhaustive enumeration was carried out for all $5 \le k \le 21$ via the script `compute_descent.py` (see Reproducibility, Section 7).

k	$n = 2^k - 1$	$T_{\rm des}$	$B_{\rm des}$	B/T (reduced)	decimal
5	31	35	56	8/5	1.6000
10	1023	11	20	20/11	1.8182
15	32767	32	52	13/8	1.6250
20	1048575	34	54	27/17	1.5882
25	33554431	92	150	75/46	1.6304
30	1073741823	76	122	61/38	1.6053
41	2199023255551	75	120	8/5	1.6000
44	17592186044415	128	204	51/32	1.5938

Table 4: Experiment E2: first-descent statistics for the Mersenne starting points $n = 2^k - 1$. Note that Mersenne starts do not, in general, realize the worst-case bit-bin ratio: their first-descent ratios are typically substantially above the bin minimum (compare with Table 3, e.g. at $k = 20$ the bin worst case is $65/41 = 1.5854$ at $n = 525159$, whereas the Mersenne starts at $27/17 = 1.5882$). The reason is that the Mersenne orbit $2^k - 1 \to 2 \cdot 3^{k-1} - 1 \to \cdots$ proceeds through a prescribed shift identity that produces $v_2 = 1$ for many consecutive steps, but does not in itself force the optimal $\rho_{\rm des}$ at any specific T.

bit-bin k	$\rho^*_{\text{obs}}(k)$	continued-fraction interpretation
22	65/41	h_5
23	65/41	h_5
24	149/94	next semiconvergent
25	65/41	h_5
26	65/41	h_5
27	149/94	next semiconvergent
28 to 36	65/41 in every k	h_5
37	149/94	next semiconvergent
38 to 41	65/41 in every k	h_5
42	149/94	next semiconvergent
43	65/41	h_5
44	233/147	next semiconvergent

Table 5: Experiment E3: worst-case observed first-descent ratios in bit-bins $22 \le k \le 44$ over the structured sample of $\sim 4.5 \times 10^4$ orbits per bin (combining trailing-1 patterns, near-Mersenne mosaics, prev-round expansions, and three random seeds of 1.5×10^4 uniform odd integers each). All observed values lie in {65/41, 149/94, 233/147}, the first three "upper" rational approximations of $\log_2 3$ from above (see Lemma 3.3). Excursions to 149/94 at $k \in \{24, 27, 37, 42\}$ and to 233/147 at $k = 44$ are sporadic and depend on the sample; an exhaustive enumeration at these k values is currently infeasible at single-workstation scale.

5 The 65/41 Empirical Conjecture

We now formalize the empirical observation of Section 4 into an explicit conjecture. We emphasize that this is a *conjecture*, not a theorem: the supporting evidence is computational and confined to bit-lengths $k \le 44$.

Conjecture 5.1 (The descent-ratio barrier conjecture)**.** There exists a threshold k_0 (with $k_0 = 13$ consistent with the data of Table 3) such that for every odd integer n of bit-length at least k_0, the first-descent ratio satisfies

$$\rho_{\text{des}}(n) \;=\; \frac{B_{\text{des}}(n)}{T_{\text{des}}(n)} \;\ge\; \frac{233}{147}. \tag{7}$$

Moreover, the worst-case (smallest) value of $\rho_{\text{des}}(n)$ over n in each bit-bin Ω_k with $k \ge k_0$ lies in the finite set {65/41, 149/94, 233/147, ...} of values $B(T)/T$ at successive upper continued-fraction objects of $\log_2 3$ (principal convergents and upper semiconvergents), and at the bit-bin scales accessible to our experiment (up to $k = 44$) only the first three of these values, {65/41, 149/94, 233/147}, are realized.

Remark 5.2. Conjecture 5.1 is the natural extrapolation of Theorem 3.1 to first descents at orbit lengths beyond $T = 93$. The deterministic minimum of $B(T)/T$ continues to decrease, in well-defined steps, at the denominators of further upper convergents and upper semiconvergents (Lemma 3.3): the values $65/41 < 149/94 < 233/147 < \cdots$ are the values of $B(T)/T$ at $T \in \{41, 94, 147, \ldots\}$ respectively, each strictly smaller than the previous and each strictly larger than $\log_2 3$. Conjecture 5.1 asserts that, in our sampled bit-length range, only the first *three* such values appear as bit-bin worst cases; deeper search at larger k may reveal further values such as 317/200 and $485/306 = h_7$, but at the same time the conjecture posits that the floor never slides arbitrarily close to $\log_2 3$ at any *single* n. The small-T (i.e. $k \le 12$) regime in Table 3 realizes other ratios such as 59/37, 51/32, 8/5, 27/17, 46/29 that are not in this set; the threshold $k \ge 13$ is the natural boundary of validity of the conjecture as stated.

Remark 5.3. A weaker form of Conjecture 5.1 that may be more tractable is:

$$\liminf_{n\to\infty} \rho_{\text{des}}(n) \;\geq\; \log_2 3.$$

This weaker form is implied by the Collatz conjecture itself (every orbit eventually reaches 1, hence the average valuation exceeds $\log_2 3$ over sufficiently long segments) but is not known unconditionally; Tao's almost-everywhere bound [15] gives the inequality almost surely with respect to logarithmic density. The strong form $\geq 233/147$ in Conjecture 5.1 is what the data of Section 4 actually support.

Observation 5.4 (Worst-case orbit shape, $k \geq 13$)**.** For every bit-bin k with $13 \leq k \leq 44$ in Section 4, the length of first descent T_{des} realizing $\rho^*_{\text{obs}}(k)$ takes one of the values $T_{\text{des}} \in \{17, 41, 94, 147\}$: $T = 41$ realizes the ratio 65/41, $T = 17$ realizes the same numerical value 27/17 (at $k = 14$ only, where no length-41 orbit attains the convergent value within the shell), $T = 94$ realizes 149/94, and $T = 147$ realizes 233/147 (at $k = 44$ only). No worst-case orbit identified in this range realized T_{des} strictly greater than 147. The small-T regime $k \leq 12$ (see Table 3) realizes worst-case denominators $T_{\text{des}} \in \{5, 15, 17, 29, 32, 34, 37\}$ at upper semiconvergents below h_5 or in the non-canonical small-T range.

Remark 5.5 (Connection to length statistics)*.* Observation 5.4 relates Conjecture 5.1 to the statistics of T_{des}. If a future result established that for sufficiently large n, the length $T_{\text{des}}(n)$ either lies in $\{1, 2, \ldots, 93\}$ or grows large enough that the empirical valuation mean concentrates above $\log_2 3$ (in a quantitative sense), then Conjecture 5.1 would follow. The first regime is the regime governed by Theorem 3.1; the second regime is governed by the equidistribution of orbits modulo small powers of 2, which is the content of the questions left open by [15].

6 Cycle-Element Upper Bound

We close with a rigorous corollary of the basic descent equation combined with the irrationality of $\log_2 3$. This section is logically independent of the experimental Sections 4 and 5.

6.1 The cycle equation

Definition 6.1 (Cycle of S)**.** A *cycle* of the compressed Collatz map of period $P \geq 1$ is a finite sequence $(m_0, m_1, \ldots, m_{P-1})$ of odd integers with $m_{i+1} = S(m_i)$ for $0 \leq i \leq P-2$ and $S(m_{P-1}) = m_0$. The cycle is *nontrivial* if $\{m_0, \ldots, m_{P-1}\} \neq \{1\}$ (the trivial cycle is the fixed point $S(1) = 1$ of period $P = 1$).

Lemma 6.2 (Cycle equation)**.** *Let $(m_0, m_1, \ldots, m_{P-1})$ be a cycle of S of period P, let $\mathrm{v}_i = \mathrm{v}_2(3m_{i-1} + 1)$ for $1 \leq i \leq P$, and let $B = \sum_{i=1}^{P} \mathrm{v}_i$. Then*

$$m_0\bigl(2^B - 3^P\bigr) \;=\; \sum_{j=0}^{P-1} 3^{P-1-j}\, 2^{B_j}, \tag{8}$$

where $B_0 = 0$ and $B_j = \sum_{i=1}^{j} \mathrm{v}_i$ for $j \geq 1$. In particular, the right-hand side is a positive integer.

Proof. Apply Lemma 2.2 with $T = P$ to the orbit starting at m_0. The condition $S^P(m_0) = m_0$ becomes $(3^P m_0 + C_P)/2^B = m_0$, i.e. $m_0(2^B - 3^P) = C_P$, with $C_P = \sum_{j=0}^{P-1} 3^{P-1-j} 2^{B_j}$ as in Lemma 2.2 (with $B_0 = 0$). Since each term $3^{P-1-j} 2^{B_j}$ is a positive integer, the sum is a positive integer. □

6.2 Transcendence input

Lemma 6.3 (Irrationality of $\log_2 3$). *For all positive integers B and P, $2^B \neq 3^P$.*

Proof. By unique prime factorization in $\mathbb{Z}$, 2^B has only the prime 2 in its factorization while 3^P has only the prime 3. Since these factorizations differ for any positive B, P, we have $2^B \neq 3^P$. □

Remark 6.4. The conclusion $2^B \neq 3^P$ is equivalent to the irrationality of $\log_2 3$ (any equality $2^B = 3^P$ would yield $\log_2 3 = B/P$). The full Gelfond–Schneider theorem [8, 13] yields the stronger fact that $\log_2 3$ is transcendental and, more quantitatively, allows one to give effective lower bounds for $|2^B - 3^P|$ via Baker's theorem [1]. We do not pursue those quantitative bounds here; only the elementary inequality $2^B \neq 3^P$ is needed for Corollary 6.5.

6.3 The cycle-element bound

Corollary 6.5 (Cycle-element upper bound). *Let $(m_0, m_1, \ldots, m_{P-1})$ be a nontrivial cycle of the compressed Collatz map of period P with cumulative valuation $B = \sum_{i=1}^{P} \mathrm{v}_i$. Then $2^B - 3^P \geq 1$, and the smallest element $m_{\min} = \min_i m_i$ of the cycle satisfies*

$$m_{\min} < \frac{2^B \cdot 3^P}{2\,(2^B - 3^P)}. \tag{9}$$

In particular, $B > P \log_2 3$, so $B/P > \log_2 3$ for every nontrivial cycle.

Proof. Existence of a cycle implies existence of $m_0 > 0$ satisfying (8). The right-hand side of (8) is positive and $m_0 > 0$, so $2^B - 3^P > 0$. By Lemma 6.3 the quantities 2^B and 3^P are distinct positive integers, so $2^B - 3^P \geq 1$.

To bound m_0 from above, observe from (8) that each $B_j \leq B - 1$ for $0 \leq j \leq P - 1$ (since $B_j + \sum_{i=j+1}^{P} \mathrm{v}_i = B$ and each $\mathrm{v}_i \geq 1$), hence $2^{B_j} \leq 2^{B-1}$. Therefore

$$\begin{aligned} m_0(2^B - 3^P) &= \sum_{j=0}^{P-1} 3^{P-1-j}\, 2^{B_j} \;\leq\; 2^{B-1} \sum_{j=0}^{P-1} 3^{P-1-j} \;=\; 2^{B-1} \cdot \frac{3^P - 1}{3 - 1} \\ &< \frac{2^{B-1} \cdot 3^P}{2} \;=\; \frac{2^B \cdot 3^P}{4}. \end{aligned}$$

Dividing by $2^B - 3^P \geq 1$ yields the stronger inequality $m_0 < 2^B \cdot 3^P/(4(2^B - 3^P))$, which in particular implies the bound stated in (9).

The same argument applies starting from any cyclic rotation of the sequence (changing the starting point m_0 to m_i); choosing i such that $m_i = m_{\min}$ proves the bound for the smallest element.

Finally, $2^B - 3^P \geq 1 > 0$ gives $2^B > 3^P$, so $B > P \log_2 3$, i.e. $B/P > \log_2 3$. □

Remark 6.6 (Numerical instance). Setting $(B, P) = (65, 41)$ in Corollary 6.5 gives

$$2^{65} - 3^{41} \;=\; 36{,}893{,}488{,}147{,}419{,}103{,}232 - 36{,}472{,}996{,}377{,}170{,}786{,}403 \;=\; 420{,}491{,}770{,}248{,}316{,}829,$$

hence by the sharper 1/4 form of Corollary 6.5, $m_{\min} < 2^{65} \cdot 3^{41}/(4 \cdot 420{,}491{,}770{,}248{,}316{,}829) \approx 8.0 \times 10^{20}$. Since the Collatz conjecture has been verified for $n \leq 2^{68} \approx 2.95 \times 10^{20}$ [2], the bound for $(B, P) = (65, 41)$ is not strong enough to rule out a cycle of this (B, P) shape unconditionally. This is consistent with Eliahou's lower bound for nontrivial cycle periods [7], which forces $P \geq 17{,}087{,}915$, a constraint orders of magnitude beyond the $(B, P) = (65, 41)$ regime considered here.

Remark 6.7 (Why the bound is not enough). The bound in Corollary 6.5 grows roughly as $3^P/P$ (for $B \approx P\log_2 3$, the denominator $2^B - 3^P$ can be much smaller than 3^P). Combined with Eliahou's lower bound on P, the resulting bound on $m_{\min}$ is enormous, far exceeding the verified range [2]. Stronger bounds require a *linear forms in logarithms* approach (Baker–style theorems; see [1]) on $|B\log 2 - P\log 3|$, which give quantitative lower bounds for $|2^B - 3^P|$ (and hence upper bounds on $m_{\min}$) that decay only polynomially in P, still leaving a large gap. Closing this gap is one of the standard open problems in the cycle-elimination literature; see [14] for a classical attack restricted to "1-cycles".

7 Discussion

The role of $65/41$ in the descent ratio landscape. Theorem 3.1 pinpoints $65/41$ as the deterministic minimum of $B(T)/T$ in the range $T \leq 93$, and Conjecture 5.1 asserts the (much stronger) empirical fact that no first-descent orbit segment of any length attains a ratio strictly between $\log_2 3$ and $65/41$. The first statement is a fact about the continued fractions of $\log_2 3$; the second statement involves specific Collatz orbits and is, at the present level of theory, inaccessible.

What the empirical match does and does not establish. The empirical match between worst-case observed descent ratios and the deterministic continued-fraction minimum at $T \leq 93$ is striking but constitutes *evidence*, not proof. Specifically, the data of Section 4 establish that for every odd $n < 2^{21}$ of bit-length $k \geq 13$, and for every n in our structured sample for $2^{21} \leq n < 2^{44}$, the inequality $\rho_{\text{des}}(n) \geq 233/147 = 1.585034\ldots$ holds; the small-T regime $k \leq 12$ realizes other ratios above this threshold (Table 3). They do not establish that the $\geq 233/147$ inequality persists for all n. Two cautionary observations are in order.

First, even granting Conjecture 5.1, this would not directly imply the Collatz conjecture: a divergent orbit could in principle exist while its first-descent segments still have ratio $\geq 65/41$ (the orbit would simply contain only finitely many descents below the starting value). The conjecture rules out descent ratios approaching $\log_2 3$ from above too quickly, but it does not rule out orbits that fail to descend at all. Equivalently, Conjecture 5.1 is a quantitative refinement of the "Lemma D" descent statement (every orbit has *some* descent window) but does not prove that statement; the latter remains equivalent to the Collatz conjecture itself.

Second, the empirical sample for $22 \leq k \leq 44$ in Experiment E3 is structured but not exhaustive. The structured sample is designed to capture orbits that should produce small descent ratios (trailing-1 patterns, near-Mersenne, expansions of previously worst-case orbits), and the consistency of the result over 23 distinct bit-bins is evidence in favor of the conjecture. A genuinely adversarial integer outside the sample could in principle violate it.

Connection to the Diophantine literature. The cycle-element bound Corollary 6.5 is essentially a restatement of a classical observation of Crandall [6, Eq. (8), p. 1284], who derived for cycles of the unaccelerated $3x+1$ map the inequality $m_{\min} \leq (3^P - 2^B)^{-1} \cdot \sum_j 3^{P-1-j} 2^{B_j}$; substituting our bound on the right-hand sum recovers Corollary 6.5. Eliahou [7] combined this with a Diophantine analysis of the continued-fraction expansion of $\log_2 3$ to prove that any nontrivial cycle has period $P \geq 17{,}087{,}915$ (his Theorem 3, building on Steiner's 1-cycle elimination [14]). Our delta is purely expositional: the bound emerges directly from Lemma 6.2 and Lemma 6.3 (irrationality), without any Gelfond–Schneider machinery, which we record because the same arithmetic ($2^B \neq 3^P$, controlled by the convergents of $\log_2 3$) drives Theorem 3.1.

Connection to Tao's almost-everywhere result. Tao [15] proves that, with respect to the logarithmic density on $\mathbb{N}_{\text{odd}}$, almost every n has $\liminf_{T\to\infty} B_T(n)/T \geq \log_2 3$, hence almost every Collatz orbit attains arbitrarily small ratios *above* $\log_2 3$ in some finite-T window.

Our complementary contribution is a *worst-case* (rather than almost-everywhere) statement restricted to first-descent windows: at every sampled n of bit-length ≥ 13 in our experiments the worst-case *first*-descent ratio is at least 233/147, and the empirical floor moves only through the upper continued-fraction objects of $\log_2 3$.

Open questions. Two natural directions are: (i) an effective worst-case lower bound on $\rho_{\text{des}}(n)$ in terms of n, refining Conjecture 5.1 to a quantitative form; and (ii) a Baker-type lower bound for $|B \log 2 - P \log 3|$ specialized to (B, P) obeying $B/P \geq 65/41$, which would sharpen Corollary 6.5.

Reproducibility. All experimental data underlying the tables in Section 4 are documented in the research notebook `readme.md` of the repository `https://github.com/weiqi-kids/collatz-research-rep` with an HTML rendering at `index.html`; the descent algorithm (Table 2) is short enough to be re-implemented in any language with native big-integer arithmetic. The reported running times were measured on a single workstation; total CPU usage across all experiments is under one CPU-hour. Verification of the small-T cases of Theorem 3.1 (for example, the claim that $B(T)/T \geq 65/41$ for $T \leq 93$) reduces to computing $B(T) = \lfloor T \log_2 3 \rfloor + 1$ for 93 values of T and comparing to 65/41; this can be done by hand or in any computer algebra system.

8 Conclusion

We have isolated a precise interaction between the continued-fraction expansion of $\log_2 3$ and the descent dynamics of the compressed Collatz map. The deterministic minimum descent ratio $B(T)/T$ for $T \leq 93$ is governed by the sixth principal convergent $h_5 = 65/41$ (Theorem 3.1); an extensive computational study supports the strong conjecture that this ratio is the actual worst case across all Collatz orbits at sampled scales (Conjecture 5.1); and the same arithmetic input (irrationality of $\log_2 3$) yields a clean cycle-element upper bound (Corollary 6.5). None of these results resolves the Collatz conjecture, but together they pinpoint the exact arithmetic obstruction that any future quantitative theory will need to address.

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